

## DIMENSIONAL REASONING

Measurements consist of two properties:

- 1) a quality or dimension, and
- 2) a quantity expressed in terms of "units"

### Dimensions

A. Everything that can be measured consists of a combination of three primitive dimensions: (introduced by Maxwell 1871)

1. Mass  $M$
2. Length  $L$
3. Time  $T$

*Note: (Temperature and electrical charge are sometimes considered as primitive dimensions, but they can be expressed in terms of  $M, L, T$ )*

B. A dimension is characteristic of the object, condition, or event and is described quantitatively in terms of defined "units".

C. Examples

1. length  $L$
2. velocity  $\frac{L}{T}$
3. force  $\frac{ML}{T^2}$

### Units provide the scale to quantify measurements

A. Measurement systems—cgs, MKS, SI—are definitions of units

B. International system of SI units

1. Primitives

- |                        |          |       |
|------------------------|----------|-------|
| a) length              | meter    | $m$   |
| b) mass                | kilogram | $kg$  |
| c) time                | second   | $s$   |
| d) elec. current       | ampere   | $A$   |
| e) luminous intensity  | candela  | $cd$  |
| f) amount of substance | mole     | $mol$ |

2. Derived units (partial list)

- |             |        |    |             |
|-------------|--------|----|-------------|
| a) force    | newton | N  | $mkg/s^2$   |
| b) energy   | joule  | J  | $m^2kg/s^2$ |
| c) pressure | pascal | Pa | $kg/(ms^2)$ |
| d) power    | watt   | W  | $m^2kg/s^3$ |

**Dimensional analysis**

## A. Fundamental rules:

1. All terms in an equation must reduce to identical primitive dimensions
2. Dimensions can be algebraically manipulated, e.g.,  $L * \frac{T}{L} = T$
3. Example:  $s = \frac{at^2}{2}$        $L = \left(\frac{L}{T^2}\right) * T^2 = L$

## B. Uses

1. Check consistency of equations
2. Deduce expressions for physical phenomena
  - a) Example: What is the period of oscillation for a pendulum?

Possible variables: length  $l$  [ $L$ ], mass  $m$  [ $M$ ],

gravity  $g$  [ $\frac{L}{T^2}$ ], i.e.,  $p = f(l, m, g)$

Period =  $T$ , so combinations of variables must be equivalent to  $T$

Only possible combination is period  $\sim \sqrt{\frac{l}{g}}$

*Note: mass is not involved*

What about an expression to relate pressure to water velocity? i.e.,  $p = f(v, ?, ?)$

- b) See Buckingham Pi Theorem

**Quantitative considerations**

## A. Each measurement carries a unit of measurement

Example: it is meaningless to say that a board is "3" long; "3" what? Perhaps "3 meters" long.

## B. Units can be algebraically manipulated (like dimensions)

## C. Conversions between measurement systems can be accommodated

through relationships between units, e.g.,  $1\text{ m} = 100\text{ cm}$ ,

$$\text{or } 100 = \frac{\text{cm}}{\text{m}}, \text{ or } \frac{1}{100} = \frac{\text{m}}{\text{cm}}$$

$$\text{Example: } 3\text{ m} = 3\text{ m} * 100 \frac{\text{cm}}{\text{m}} = 3 * 100\text{ m} * \frac{\text{cm}}{\text{m}} = 300\text{ cm}$$

- D. Arithmetic manipulations between terms can take place only with identical units

$$\text{Example: } 3\text{ m} + 2\text{ cm} = ? \text{ but, } 3\text{ m} * 100 * \frac{\text{cm}}{\text{m}} + 2\text{ cm} = 302\text{ cm}$$

### "Dimensionless" Quantities

- A. Dimensional quantities can be made "dimensionless" by "normalizing" them with respect to another dimensional quantity of the same dimensionality.

Example: a length measurement error  $DX (m)$  can be made "dimensionless" by dividing by the measured length  $X (m)$ , so  $DX' = DX (m) / X (m)$ .  $DX'$  has no dimensions. (In this case the result is a percentage).

- B. Equations and variables can be made dimensionless

- C. Useful properties:

1. dimensionless equations and variables are independent of units.
2. relative importance of terms can be easily estimated
3. scale (battleship or model ship) is automatically built into the dimensionless expression
4. reduces many problems to a single problem through normalization, e.g.,  $\frac{x - \mu}{\sigma}$ , to obtain a Gaussian distribution  $N(0,1)$

- D. A non-dimensional example: Newton's drag law

$$m[M] \frac{dv}{dt} [LT^{-2}] = -m[M]g[LT^{-2}] - \alpha[ML^{-1}]v^2 [L^2T^{-2}]$$

i.e., the rate of change of momentum of an object with mass  $m$  equals the force due to gravity minus a viscous drag force proportional to velocity  $v^2$ .

Construct characteristic mass, length, and time scales as

$$m_c = m \quad l_c = m / \alpha \quad t_c = \left( \frac{m}{\alpha g} \right)^{1/2}$$

and develop a characteristic velocity scale as  $v_c = \frac{l_c}{t_c} = \left( \frac{mg}{\alpha} \right)^{1/2}$ .

In terms of these characteristic variables (with dimensions M, L, T, and L/T, respectively), define new non-dimensional variables as follows:

$$m' = m / m_c \quad t' = t / t_c \quad v' = v / v_c$$

Since  $\frac{d}{dt} = \frac{dt'}{dt} \frac{d}{dt'}$  we can rewrite our original equation in terms of the non-dimensional variables as:

$$m_c m' \frac{1}{t_c} \frac{d}{dt'} (v' v_c) = -m_c m' g - \alpha v'^2 v_c^2 \Rightarrow m_c m' \left( \frac{\alpha g}{m_c m'} \right)^{1/2} \left( \frac{m_c m' g}{\alpha} \right)^{1/2} \frac{dv'}{dt'} = -m_c m' g - \frac{\alpha m_c m' g}{\alpha} v'^2$$

Canceling things out gives  $\frac{dv'}{dt'} = -1 - v'^2$

Note that, in this case, all the parameters have disappeared in this dimensionless formulation. We can deduce that viscous drag is unimportant when  $v' \ll 1$ —that is when  $v \ll (mg/\alpha)^{1/2}$ .

E. A more complicated dimensional problem is as follows: Suppose

want to analyze the lift force  $f \left[ \frac{ML}{T^2} \right]$  of an airfoil. We

presume it to depend on area  $A \left[ L^2 \right]$ , velocity  $v \left[ \frac{L}{T} \right]$ , density

of air  $\rho \left[ \frac{M}{L^3} \right]$ , and viscosity of air  $\mu \left[ \frac{M}{LT} \right]$ . An

answer is

$$f \sim \rho v^2 A$$

But it could also be

$$f \sim \mu v A^{\frac{1}{2}}$$

What to do?

### Buckingham Pi Theorem (Buckingham 1915)

- A. Pi theorem tells how many dimensionless groups define a problem  
 B. Theorem: If  $n$  variables are involved in a problem and these are expressed using  $k$  primitive dimensions, then  $(n-k)$  dimensionless groups are required to characterize the problem.

Example: in the pendulum, the variables were time  $[T]$ , gravity  $\left[\frac{L}{T^2}\right]$ , length  $[L]$ , mass  $[M]$ . So,  $n = 4$   $k = 3$ . So, only one dimensionless group  $t\left(\frac{l}{g}\right)^{.5}$  describes the system.

Example: in the airfoil problem, we have force, area, velocity, density, and viscosity, i.e.,  $n = 5$ ; and they involve  $M, L, T$ , so  $k = 3$ . So two dimensionless groups are required.

C. The functional groups are related as

$$\pi_1 = f(\pi_2, \pi_3, \dots, \pi_{n-k})$$

D. How to find the dimensional groups:

1. Pendulum example

$$\pi_1 = t^a l^b g^c m^d \quad \text{where } a, b, c, d \text{ are coefficients to be determined}$$

In terms of dimensions:

$$\begin{aligned} T^a L^b (LT^{-2})^c M^d &= M^0 L^0 T^0 \\ T^{(a-2c)} L^{(b+c)} M^d &= M^0 L^0 T^0 \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad a - 2c &= 0 \\ b + c &= 0 \\ d &= 0 \end{aligned}$$

Three equations, four unknowns: not unique. Arbitrarily choose  $a = 1$ . Then  $c = 1/2$ ,  $b = -1/2$ ,  $d = 0$ . This yields

$$\pi_1 = t \sqrt{\frac{g}{l}} = c$$

2. Airfoil lift example

$$\pi_1 = F^a A^b v^c \rho^d \mu^e$$

Dimensionally

$$\left(\frac{ML}{T^2}\right)^a (L^2)^b \left(\frac{L}{T}\right)^c \left(\frac{M}{L^3}\right)^d \left(\frac{M}{LT}\right)^e = M^{(a+d+e)} L^{(a+2b+c-3d-e)} T^{(-2a-c-e)} = M^0 L^0 T^0$$

$$\begin{aligned} \text{So} \quad a + d + e &= 0 \\ a + 2b + c - 3d - e &= 0 \\ -2a - c - e &= 0 \end{aligned}$$

Three equations, five unknowns: not unique; Two arbitrary choices. Choose  $a=1$  to keep  $F$ , choose  $e=0$  to eliminate  $\mu$ .

$$\text{This gives } \pi_1 = FA^{-1}v^{-2}\rho^{-1}\mu^0 = \frac{F}{(\rho v^2 A)}$$

Now choose  $a=0$  to eliminate  $F$ ,  $c=1$  to include  $v$ .

$$\text{This gives } \pi_2 = \rho^1 v^1 A^{\frac{1}{2}} \mu^{-1} = \frac{\rho v A^{\frac{1}{2}}}{\mu}$$

$$\text{Finally, } \pi_1 = f(\pi_2) \quad \text{or} \quad \frac{F}{\rho v^2 A} = f\left(\frac{\rho v A^{\frac{1}{2}}}{\mu}\right)$$

Note: In this particular case  $\pi_1$  is called the lift coefficient  
 $\pi_2$  is called the Reynolds number

If parameters are changed, but  $\pi_2$  remains constant, then  $\pi_1$  would also remain constant

3. Example: the period of oscillation of a small drop of liquid under the influence of surface tension.

Assuming no influence of gravity and a restoring force dependent only on surface tension, the period of oscillation  $t$  [T] will depend only on surface tension  $\sigma$  [MT<sup>-2</sup>], density  $\rho$  [ML<sup>-3</sup>], and diameter [L].

$$\text{Answer: } t = C \sqrt{\frac{\rho d^3}{\sigma}}$$

### Scaling, modeling, similarity

1. Types of “similarity” between two objects/processes
  - a) Geometric similarity—linear dimensions are proportional; angles are the same
  - b) Kinematic similarity—includes proportional time scales, i.e., velocity, are similar.
  - c) Dynamic similarity—includes force scale similarity, i.e., equality of Reynolds number (inertial/viscous), Froude number (inertial/buoyancy), Rossby number (inertial/Coriolis), Euler number (inertial/surface tension)

Movies often use models for scenes involving ships, explosions, crashes, etc. Sometimes the dynamic similarity is close enough to appear real.

## 2. Distorted models

- a) Sometimes it's necessary to violate geometric similarity: A 1/1000 scale model of the Chesapeake Bay is ten times as deep as it should be, because the real Bay is so shallow that, with proportional depths, the average model depth would be 6mm. Too shallow to exhibit stratified flow.

## 3. Scaling

- a) What's the biggest elephant? If one tries to keep similar geometric proportions, weight  $\propto L^3$ , where L is a characteristic length, say height. However, an elephant's ability to support his weight is proportional to the cross-sectional area of his bones, say  $R^2$ . Therefore, if his height doubles, his bones would have to increase in radius as  $2\sqrt{2} R$ , not  $2R$ . [Note: A cross-section of  $8 R^2 = (2\sqrt{2} R)^2$ ]. So, with increasing size, an elephant will eventually have legs whose cross-sectional area will extend beyond its body.