

The unit sample and unit step

Let's examine some special signals, first in discrete time, then in continuous time.

Definition 1.1. *The discrete time unit step is given by*

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

The unit sample or impulse is defined as

$$\delta[n] = u[n] - u[n - 1]$$

We notice that they are related via the sum relation

$$u[n] = \sum_{k=-\infty}^n \delta[k]$$

Notice the unit sample sifts signals

Proposition 1.1. *The unit sample has the “sampling property,” picking off values of signals that it sums against:*

$$x[n] = \sum_k x[n - k] \delta[k]$$

This is true for all signals, implying we can derive various properties, the “summed” and “differenced” versions. Defining $x[n] = u[n]$, then

$$u[n] = \sum_{k=-\infty}^n \delta[k]$$

Now let's examine linearity and time invariance.

Example 1.1. *Define the system with input $x[n]$ and output $y[n]=nx[n]$. This system is linear but not time invariant.*

To see linearity is straightforward, Take linear combinations of inputs and verify outputs are linear combinations. To see the system is not time invariant, define input $x_1[n] = \delta[n]$, then output is $y_1[n] = n\delta[n] = 0$ for all n . Now shift the input, $x_2[n] = \delta[n - 1]$. But for this $y_2[1] = 1$.

Definition 1.2. *The continuous time version has a similar form:*

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The differentiated for is the unit impulse:

$$\delta(t) = \dot{u}(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u(t + \epsilon) - u(t - \epsilon))$$

Proposition 1.2. *The sifting property follows for smooth functions:*

$$x(t) = \int x(t - \tau) \delta(\tau) d\tau$$

To see this, we can argue loosely swapping the order of the limit according to

$$\begin{aligned} x(t) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int x(t - \tau) (u(\tau + \epsilon) - u(\tau - \epsilon)) d\tau \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} x(t - \tau) d\tau = x(t) \end{aligned}$$

In particular, defining $x(t)=u(t)$, then we see the integrated property follows

$$\begin{aligned} u(t) &= \int_{\sigma=-\infty}^t \delta(\sigma) d\sigma \\ \delta(t) &= \dot{u}(t) \end{aligned}$$