Properties: Time Invariance

There is a second property which this first order differential equation enjoys, the time invariance property coming from the constant coefficient properties. To the time invariance, simply stated, an input shifted results in an output shifted.

Definition 2.1. *A system with input x(t) and output y(t) is time-invariant if $x(t-t_0)$ creates output $y(t-t_0)$, for all inputs x and shifts t_0.*

Let us see that this is true for our differential equation for the special complex exponentials. Choose $x(t)=e^{i\omega t}$ with output $y(t)=\frac{1}{a+i\omega}e^{i\omega t}$, then $x(t-t_0)=e^{i\omega t}e^{-i\omega t_0}$. The output becomes

$$e^{-i\omega t}\frac{1}{a+i\omega}e^{i\omega t}=\frac{1}{a+i\omega}e^{i\omega(t-t_0)}=y(t-t_0)$$

Thus we see that a simple shift of the complex exponential, the output is shifted in the same manner. Demonstrate to yourself that the same is true for arbitrary periodic inputs represented via the Fourier series $x(t)=\sum_k a_k e^{i\omega_k t}$.

Properties: Commutative, Causality, BIBO Stability

Proposition 1.4. *Convolution has the property that*

$$x(t) ** h(t) = h(t) ** x(t)$$

$$x[k] ** h[k] = h[k] ** x[k]$$

Show this by redefining the limits of the integral.

Definition 1.4. *A system is causal, if the output y(t) at time t is not a function of future inputs.*

If the system is causal, then this implies h(t)=0, t < 0. Alternatively, h[n]=0, n < 0. To see this,

$$y(t) = x(t) ** h(t) = h(t) ** x(t) = \int x(\sigma)h(t-\sigma)d\sigma$$

$$= f(x(s), s > t), \text{ if h(t)} \neq 0 \text{ for t} < 0$$

Definition 1.5. *A system is BIBO stable (bounded-input, bounded output), if a bounded input implies a bounded output.*

Here is a sufficient condition:

Proposition 1.5. *If $\int |h(t)|dt < \infty$ then the system is BIBO stable.*

Let $|x(t)| < B$, ∀t, then

$$y(t) = x(t) ** h(t) = h(t) ** x(t) = \int h(\sigma)x(t-\sigma)d\sigma$$

$$\leq B \int |h(\sigma)| d\sigma < C$$

Example 1.6. *Let h(t) = u(t+3), then the system is not causal.*

$$y(t) = x(t) ** h(t) = \int x(\sigma) u(t + 3 - \sigma) d\sigma = \int_0^t x(\sigma) d\sigma$$

$$= f(x(s), s > t)$$

Let h(t) = u(t), then system not BIBO stable. Let input x(t)=u(t), then $|x(t)| \leq 1$ *bounded, and*

$$y(t) = x(t) ** u(t) = \int u(\sigma) u(t - \sigma) d\sigma = \int_0^t d\sigma = t$$

There does not exist a bound C for all time t.