Complex Exponentials are Eigenfunctions of LTI Systems

Proposition 1.6. *For input $x(t) = e^{i\omega t}$ with LTI system with impulse response h(t) and output y(t), then,*

$$y(t) = e^{i\omega t} \int h(\sigma) e^{-i\omega\sigma} d\sigma$$

For input $x[n] = e^{i\omega n}$ and discrete LTI system then output

$$y[n] = e^{i\omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-i\omega k}$$

To prove it, simply do the continuous time:

$$y(t) = x(t) ** h(t) = h(t) ** x(t) = \int h(\sigma) x(t-\sigma) d\sigma$$

$$= \int h(\sigma) e^{i\omega(t-\sigma)} d\sigma = e^{i\omega t} \int h(\sigma) e^{-i\omega\sigma} d\sigma$$

Example 1.7. *Given the differential equation*

$$x(t) = \dot{y}(t) + \alpha y(t)$$

Then $h(t) = e^{-\alpha t} u(t)$ giving

$$\int_{-\infty}^{\infty} h(\sigma) e^{-i\omega\sigma} d\sigma = \int_{0}^{\infty} e^{-(\alpha+i\omega)\sigma} d\sigma$$

$$= -\frac{e^{-(\alpha+i\omega)}\Big|_0^\infty}{a + i\omega}$$

For $\alpha > 0$, then the upper limit is zero and the result becomes

$$\int_0^\infty e^{-(\alpha+i\omega)\sigma} d\sigma = \frac{1}{a+i\omega}$$

$$x[n] = y[n] - \alpha y[n-1]$$