

2.1 The Convolution Integral

So now we have examined several simple properties that the differential equation satisfies linearity and time-invariance. We have also seen that the complex exponential has the special property that it passes through unchanged only by a complex number the differential equation. Also, we have discussed the role of transforms, as representing arbitrary inputs via the superpositions of complex exponentials. This discussion is often called a "frequency domain analysis". Frequency domain analysis studies the outputs of linear and time-invariant systems via their response to complex exponentials. Now we turn our focus to a pure time domain analysis, understanding the response of the differential equation directly in terms of its time domain inputs. For this we explore the "convolution integral". We do this by solving the first-order differential equation directly using integrating factors. For this, examine the differential equation and introduce the integrating factor $f(t)$ which has the property that it makes one side of the equation into a total differential. Define

$$\begin{aligned} f(t)x(t) &= f(t)\dot{y}(t) + f(t)ay(t) \\ &= \frac{d}{dt}(f(t)y(t)) \end{aligned}$$

which implies

$$\dot{f}(t)y(t) + f(t)\dot{y}(t) = f(t)\dot{y}(t) + af(t)y(t)$$

This implies the integrating factor is $f(t) = e^{at}$, and using the boundary condition $y(-\infty) = 0$ the total differential is solved giving

$$e^{at}y(t) = \int_{-\infty}^t e^{a\sigma}x(\sigma)d\sigma$$

We have almost arrived at our convolution formula. For this introduce the unit step function, and the definition of the convolution formulation. The unit-step function is zero to the left of the origin, and 1 elsewhere:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Definition 2.2. Given time signals $f(t)$, $g(t)$, then their convolution is defined as

$$f(t) ** g(t) = \int_{-\infty}^{\infty} f(\sigma)g(t - \sigma)d\sigma$$

Proposition 2.1. The output of this first order differential equation with input $x(t)$ is given according to

$$y(t) = x(t) ** e^{-at}u(t)$$

To see this, simply use the property of the unit step to rewrite the solution of Eqn. 13 according to

$$y(t) = \int_{-\infty}^{\infty} x(\sigma)e^{-a(t-\sigma)}u(t-\sigma)$$

We make the following comment. Notice the output is a function of the input “convolved” with a property of the system, $e^{-at}u(t)$. This property we will call the “impulse response” of the system and we will study it extensively. For LTI systems this will always be true, although the property of the system will change depending on the system. So we have arrived at the second major component of our study of linear, time-invariant systems. To understand the outputs of LTI systems to arbitrary inputs, one needs to understand the convolution integral. The remaining 12 lectures work to generalize and strengthen these very notions.